NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, NATURAL RESOURCES AND APPLIED SCIENCES
SCHOOL OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

QUALIFICATION: Bachelor of Science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BSAM	**LEVEL:** 6
COURSE CODE: PBT602S	**COURSE NAME**: Probability Theory 2
SESSION: JUNE 2023	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	Dr D. B. GEMECHU
MODERATOR:	Prof R. KUMAR

INSTRUCTIONS
1. There are 5 questions, answer ALL the questions by showing all the necessary steps. 2. Write clearly and neatly. 3. Number the answers clearly. 4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculators with no cover.

THIS QUESTION PAPER CONSISTS OF 3 PAGES (Including this front page)

Question 1 [12 marks]

1.1. Define the following terms:

1.1.1. Power set, $\mathcal{P}(S)$ [2]

1.1.2. Sigma algebra, $\sigma(S)$ [2]

1.1.3. Boolean algebra, $\mathfrak{B}(S)$ [2]

1.2. Consider an experiment of rolling a die with four faces once.

1.2.1. Find the power set of the sample space S for this experiment, where S represents the sample space for a random experiment of rolling a die with six faces. [3]

1.2.2. Show that the set $\sigma(X) = \{\phi, S, \{2,3\}, \{1,4\}\}$ is a sigma algebra. [3]

Question 2 [27 marks]

2.1. Let X be a continuous random variable with p.d.f. given by

$$f_X(x) = \begin{cases} x & if\, 0 < x < 1 \\ 2 - x & if\, 1 \leq x < 2 \\ 0 & otherwise \end{cases}$$

Then find cumulative density function of X [7]

2.2. The cumulative distribution function (c.d.f.) of a random variable X is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{4} & \text{for } 0 \leq x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

Then use the c.d.f. of X to find

2.2.1. $P(2 < X \leq 3)$ [2]

2.2.2. $P(X \geq 1.5)$ [1]

2.2.3. the 25th percentile value of X. [2]

2.3. Consider the following joint p.d.f. of X and Y.

$$f(x, y) = 3(x + y)I_{(0,1)}(x + y)I_{(0,1)}(x)I_{(0,1)}(y)$$

Find the marginal p.d.f. of Y. [4]

2.4. Let X and Y be a jointly distributed continuous random variable with joint p.d.f. of

$$f_{XY}(x, y) = \begin{cases} 1.2(x + y^2) & for\, 0 \leq x \leq 1 and\, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

2.4.1. Show that marginal pdf of X, $f_X(x) = \frac{6}{5}\left(x + \frac{1}{3}\right) I_{(0,\ 1)}(x)$. [2]

2.4.2. Find the conditional distribution of Y given $X = \frac{1}{4}$. [3]

2.4.3. Find $P(Y \geq 0.15 | X = 0.25)$. [3]

2.4.4. Find the conditional mean Y given $X = \frac{1}{4}$. [3]

Question 3 [24 marks]

3.1. Let X and Y be two random variables and let $a,\ b,\ c\ and\ k$ be any constant numbers. Then $Cov(aX + c, bY + k) = \text{abCov(X, Y)}$. [5]

3.2. Let Y_1, Y_2, and Y_3 be three random variables with $E(Y_1) = 5$, $E(Y_2) = 12$, $E(Y_3) = 4$, $\sigma_{Y_1}^2 = 2$, $\sigma_{Y_2}^2 = 3$, $\sigma_{Y_3}^2 = 1$, $\sigma_{Y_1Y_2} = -0.6$, $\sigma_{Y_1Y_3} = 0.3$, and $\sigma_{Y_2Y_3} = 2$. If $R = 2Y_1 - 3Y_2 + Y_3$, then find

3.2.1. the expected value of R. [2]

3.2.2. the correlation coefficient between Y_1 and Y_3 and comment on your result. [3]

3.2.3. the variance of R. [5]

3.3. The joint probability density function of the random variables X, Y, and Z is

$$f(x,y,z) = \begin{cases} \frac{4}{9}xyz^2, & 0 < x < 1; 0 < y < 1; 0 < z < 3, \\ 0, & elsewhere. \end{cases}$$

Find the joint marginal density function of Y and Z. **Hint**: find $f_{YZ}(y,z)$. [4]

3.4. If X_1, X_2, and X_3 are **DISCRETE** random variables with joint p.m.f. $f(x_1, x_2, x_3)$, then for any constants c_1, c_2 and c_3, show that $E\left(\sum_{i=1}^{3} c_i X_i\right) = \sum_{i=1}^{3} c_i E(X_i)$. [5]

QUESTION 4 [17 marks]

4.1. Suppose that X is a random variable having a binomial distribution with the parameters n and p (i.e., $X \sim Bin(n, p)$).

4.1.1. Show that the moment generating function of X is given by $M_X(t) = \left(1 - p(1 - e^t)\right)^n$. **Hint**: $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. [4]

4.1.2. Find the cumulant generating function of X and hence find the first cumulant. [5]

4.2. Let the random variables $X_k \sim Poisson(\lambda_k)$ for $k = 1, \ldots, n$ be independent Poisson random variables. If we define another random variable $Y = X_1 + X_2 + \cdots + X_n$, then find the characteristics function of Y, $\phi_Y(t)$. Comment on the distribution of Y based on your result. [**Hint** $\phi_{X_k}(t) = e^{\lambda_k(e^{it} - 1)}$]. [8]

QUESTION 5 [20 marks]

5.1. Suppose that X and Y are independent, continuous random variables with densities $f_X(x)$ and $f_Y(y)$. If $Z = X + Y$, then show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy.$$ [5]

5.2. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. Use the convolution formula to show that $X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. [7]

5.3. Let X_1 and X_2 have joint p.d.f. $f(x_1, x_2) = 2e^{-(x_1 + x_2)}$ for $0 < x_1 < x_2 < 1$. Let $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Find the joint p.d.f. of Y_1 and Y_2, $g(y_1, y_2)$. [8]

=== END OF PAPER===

TOTAL MARKS: 100